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Conditional Identities Involving the Angles of a Triangle

Before we begin, let us first recall the following formulae:

(i) $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(ii) $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

(iii) $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(iv) $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

TYPE 1 IDENTITIES INVOLVING SINES AND COSINES

METHOD

1. Express the sum of the first two terms as a product.
2. In this product, express the sum of two angles in terms of the third angle, using $A + B + C = \pi$.
3. Expand the third term by using one of the following relations:
   $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = (2\cos^2\theta - 1) = (1 - 2\sin^2\theta)$.
4. Take the common factor outside.
5. Express the T-ratio of a single angle into a sum of two angles, and use the necessary formula from the ones given in the box above.

SOLVED EXAMPLES

EXAMPLE 1 *If $A + B + C = \pi$, prove that*

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$$

SOLUTION We have

$$\begin{aligned} \text{LHS} &= (\sin 2A + \sin 2B) + \sin 2C \\ &= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C \quad \text{[by formula]} \\ &= 2\sin(\pi - C)\cos(A-B) + 2\sin C\cos C \\ &= 2\sin C\cos(A-B) + 2\sin C\cos C \quad [\because \sin(\pi - C) = \sin C] \end{aligned}$$

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$= 2\sin C[\cos(A-B) + \cos C]$

$= 2\sin C[\cos(A-B) + \cos\{\pi-(A+B)\}]$ $[\because C = \pi-(A+B)]$

$= 2\sin C[\cos(A-B) - \cos(A+B)]$ $[\because \cos(\pi-\theta) = \cos\theta]$

$= 2\sin C[2\sin A \sin B]$

$= 4\sin A \sin B \sin C = \text{RHS}.$

$\therefore$ $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$

EXAMPLE 2 *If $A + B + C = \pi$, prove that*

$$\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C.$$

SOLUTION We have

LHS $= (\sin 2A - \sin 2B) + \sin 2C$

$= 2\cos(A+B)\sin(A-B) + 2\sin C\cos C$ [by formula]

$= 2\cos(\pi - C)\sin(A-B) + 2\sin C\cos C$

$= -2\cos C\sin(A-B) + 2\sin C\cos C$ $[\because \cos(\pi - C) = -\cos C]$

$= -2\cos C[\sin(A-B) - \sin C]$

$= -2\cos C[\sin(A-B) - \sin\{\pi-(A+B)\}]$ $[\because C = \pi-(A+B)]$

$= -2\cos C[\sin(A-B) - \sin(A+B)]$

$= -2\cos C[-2\cos A \sin B]$

$= 4\cos A \sin B \cos C = \text{RHS}.$

$\therefore$ $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C.$

EXAMPLE 3 *If $A + B + C = \pi$, prove that*

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C.$$

SOLUTION We have

LHS $= \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$

$= 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1$

$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$

$= -2\cos C[\cos(A-B) - \cos C] - 1$

$= -1 - 2\cos C[\cos(A-B) - \cos\{\pi-(A+B)\}]$

$= -1 - 2\cos C[\cos(A-B) + \cos(A+B)]$

$= -1 - 2\cos C[2\cos A \cos B]$

$= -1 - 4\cos A \cos B \cos C = \text{RHS}.$

$\therefore$ $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C.$

EXAMPLE 4 *If $A + B + C = \pi$, prove that*

$$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C.$$

SOLUTION We have

LHS $= (\cos 2A + \cos 2B) - \cos 2C$

$= 2\cos(A+B)\cos(A-B) - 2\cos^2 C + 1$

$$= 2\cos(\pi - C)\cos(A - B) - 2\cos^2 C + 1$$
$$= -2\cos C\cos(A - B) - 2\cos^2 C + 1$$
$$= 1 - 2\cos C[\cos(A - B) + \cos C]$$
$$= 1 - 2\cos C[\cos(A - B) + \cos\{\pi - (A + B)\}]$$
$$= 1 - 2\cos C[\cos(A - B) - \cos(A + B)]$$
$$= 1 - 2\cos C[2\sin A \sin B]$$
$$= 1 - 4\sin A \sin B \cos C = \text{RHS}.$$

$\therefore \quad \cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C.$

EXAMPLE 5 *If $A + B + C = \pi$, prove that*

$$\cos 4A + \cos 4B + \cos 4C = -1 + 4\cos 2A \cos 2B \cos 2C.$$

SOLUTION We have

$$\text{LHS} = \cos 4A + \cos 4B + \cos 4C$$
$$= 2\cos(2A + 2B)\cos(2A - 2B) + \cos 4C$$
$$= 2\cos(2\pi - 2C)\cos(2A - 2B) + (2\cos^2 2C - 1)$$
$$[\because A + B + C = \pi \Rightarrow (2A + 2B) = (2\pi - 2C)]$$
$$= 2\cos 2C\cos(2A - 2B) + 2\cos^2 2C - 1$$
$$= 2\cos 2C[\cos(2A - 2B) + \cos 2C] - 1$$
$$= 2\cos 2C[\cos(2A - 2B) + \cos\{2\pi - (2A + 2B)\}] - 1$$
$$[\because 2A + 2B + 2C = 2\pi \Rightarrow 2C = 2\pi - (2A + 2B)]$$
$$= 2\cos 2C[\cos(2A - 2B) + \cos(2A + 2B)] - 1$$
$$= 2\cos 2C[2\cos 2A \cos 2B] - 1$$
$$= -1 + 4\cos 2A \cos 2B \cos 2C = \text{RHS}.$$

$\therefore \quad \cos 4A + \cos 4B + \cos 4C = -1 + 4\cos 2A \cos 2B \cos 2C.$

EXAMPLE 6 *If $A + B + C = \pi$, prove that*

$$\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

SOLUTION We have

$$\text{LHS} = (\sin A + \sin B) - \sin C$$
$$= 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$= 2\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}\left[\because \frac{A + B}{2} = \frac{\pi}{2} - \frac{C}{2}\right]$$
$$= 2\cos\frac{C}{2}\cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A - B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left\{\frac{\pi}{2} - \left(\frac{A+B}{2}\right)\right\}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \text{RHS}.$$

$$\therefore \quad \sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

<u>EXAMPLE 7</u> *If $A + B + C = \pi$, prove that*

$$\cos A + \cos B - \cos C = \left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1.$$

SOLUTION We have

$$\text{LHS} = (\cos A + \cos B) - \cos C$$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \left(1 - 2\sin^2\frac{C}{2}\right)$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 1 + 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right] - 1$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\left\{\frac{\pi}{2} - \frac{(A+B)}{2}\right\}\right] - 1$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right] - 1$$

$$= 2\sin\frac{C}{2}\left\{2\cos\frac{A}{2}\cos\frac{B}{2}\right\} - 1$$

$$= \left\{4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right\} - 1 = \text{RHS}.$$

$$\therefore \quad \cos A + \cos B - \cos C = \left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1.$$

TYPE 2 IDENTITIES INVOLVING SQUARES OF SINES AND COSINES

METHOD Change the squares of sines and cosines into cosines of double the angles by using the formulae

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \text{ and } \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

Now, proceed as above.

<u>EXAMPLE 8</u> *If $A + B + C = \pi$, prove that*

$$\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A\cos B\cos C).$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \sin^2 A + \sin^2 B + \sin^2 C\\ &= \frac{1}{2}(1 - \cos 2A) + \frac{1}{2}(1 - \cos 2B) + \frac{1}{2}(1 - \cos 2C)\\ &= \frac{3}{2} - \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C)\\ &= \frac{3}{2} - \frac{1}{2}(-1 - 4\cos A \cos B \cos C) \quad \text{[see Example 3]}\\ &= 2 + 2\cos A \cos B \cos C = 2(1 + \cos A \cos B \cos C) = \text{RHS}.\end{aligned}$$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C)$.

EXAMPLE 9 *If $A + B + C = \pi$, prove that*

$$\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C.$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos^2 A + \cos^2 B - \cos^2 C\\ &= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) - \frac{1}{2}(1 + \cos 2C)\\ &= \frac{1}{2} + \frac{1}{2}(\cos 2A + \cos 2B - \cos 2C)\\ &= \frac{1}{2} + \frac{1}{2}(1 - 4\sin A \sin B \cos C) \quad \text{[see Example 4]}\\ &= 1 - 2\sin A \sin B \cos C = \text{RHS}.\end{aligned}$$

$\therefore \quad \cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$.

EXAMPLE 10 *In a $\triangle ABC$, prove that*

$$\cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right) = \frac{3}{2}\cdot$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right)\\ &= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}\left\{1 + \cos\left(2A + \frac{2\pi}{3}\right)\right\} + \frac{1}{2}\left\{1 + \cos\left(2A - \frac{2\pi}{3}\right)\right\}\\ &= \frac{3}{2} + \frac{1}{2}\cos 2A + \frac{1}{2}\left\{\cos\left(2A + \frac{2\pi}{3}\right) + \cos\left(2A - \frac{2\pi}{3}\right)\right\}\\ &= \frac{3}{2} + \frac{1}{2}\cos 2A + \cos 2A \cos\frac{2\pi}{3}\\ &\qquad [\because \cos(A + B) + \cos(A - B) = 2\cos A \cos B]\\ &= \frac{3}{2} + \frac{1}{2}\cos 2A - \frac{1}{2}\cos 2A \quad \left[\because \cos\frac{2\pi}{3} = -\frac{1}{2}\right]\\ &= \frac{3}{2} = \text{RHS}.\end{aligned}$$

EXAMPLE 11 *If* $A + B + C = \pi$, *prove that*

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} \\
&= \frac{1}{2}(1 - \cos A) + \frac{1}{2}(1 - \cos B) - \frac{1}{2}(1 - \cos C) \\
&= \frac{1}{2} - \frac{1}{2}(\cos A + \cos B - \cos C) \\
&= \frac{1}{2} - \frac{1}{2}\left[\left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1\right] \quad \text{[see Example 7]} \\
&= 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \text{RHS}.
\end{aligned}$$

$$\therefore \quad \sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

EXAMPLE 12 *If* $A + B + C = \pi$, *prove that*

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} \\
&= \frac{1}{2}(1 + \cos A) + \frac{1}{2}(1 + \cos B) - \frac{1}{2}(1 + \cos C) \\
&= \frac{1}{2} + \frac{1}{2}(\cos A + \cos B - \cos C) \\
&= \frac{1}{2} + \frac{1}{2}\left[\left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1\right] \\
&= 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \text{RHS}.
\end{aligned}$$

$$\therefore \quad \cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

EXAMPLE 13 *If* $A + B + C = \pi$, *prove that*

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \\
&= \frac{1}{2}(1 + \cos A) + \frac{1}{2}(1 + \cos B) + \frac{1}{2}(1 + \cos C) \\
&= \frac{3}{2} + \frac{1}{2}(\cos A + \cos B + \cos C)
\end{aligned}$$

$$= \frac{3}{2} + \frac{1}{2}\left[2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C\right]$$

$$= \frac{3}{2} + \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + \frac{1}{2}\left(1 - 2\sin^2\frac{C}{2}\right)$$

$$\left[\because \left(\frac{A+B}{2}\right) = \left(\frac{\pi}{2} - \frac{C}{2}\right) \text{ and } \cos C = 1 - 2\sin^2\frac{C}{2}\right]$$

$$= 2 + \sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - \sin^2\frac{C}{2}$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left\{\frac{\pi}{2} - \left(\frac{A+B}{2}\right)\right\}\right]$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2 + \sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$= 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = \text{RHS.}$$

$$\therefore \quad \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

TYPE 3 IDENTITIES INVOLVING TANGENTS

METHOD

1. Using $A + B + C = \pi$, express the sum of two angles in terms of the third.
2. Take tangents on both sides and expand the LHS.
3. Cross multiply and transpose.

Similarly, we proceed for identities involving cotangents.

EXAMPLE 14 *If* $A + B + C = \pi$, *prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

SOLUTION Here, $A + B + C = \pi$

$\Rightarrow \quad (A + B) = (\pi - C)$

$\Rightarrow \quad \tan(A + B) = \tan(\pi - C)$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C.$

EXAMPLE 15 *If* $A + B + C = \pi$, *prove that*

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

SOLUTION Here, $A+B+C=\pi$

$$\Rightarrow \left(\frac{A}{2}+\frac{B}{2}\right)=\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$$

$$\Rightarrow \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

EXAMPLE 16 *If $A+B+C=\pi$, prove that*

$$\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$$

SOLUTION Here, $A+B+C=\pi$

$$\Rightarrow (A+B)=(\pi-C)$$

$$\Rightarrow \cot(A+B)=\cot(\pi-C)=-\cot C$$

$$\Rightarrow \frac{\cot A\cot B-1}{\cot A+\cot B}=-\cot C$$

$$\Rightarrow \cot A\cot B-1=-\cot C\cot A-\cot B\cot C$$

$$\Rightarrow \cot B\cot C+\cot C\cot A+\cot A\cot B=1.$$

EXAMPLE 17 *If $A+B+C=\pi$, prove that*

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

SOLUTION Here, $A+B+C=\pi$

$$\Rightarrow \left(\frac{A}{2}+\frac{B}{2}\right)=\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)=\tan\frac{C}{2}$$

$$\Rightarrow \frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\frac{1}{\cot\frac{C}{2}}$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}-\cot\frac{C}{2}$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

EXERCISE 16

If $A+B+C=\pi$, prove that

1. $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$
2. $\cos 2A - \cos 2B - \cos 2C = -1 + 4\cos A \sin B \sin C$
3. $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$
4. $\sin A + \sin B + \sin C = 4\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
5. $\cos A + \cos B + \cos C = 1 + 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
6. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
7. $\sin (B+C-A) + \sin (C+A-B) - \sin (A+B-C) = 4\cos A \cos B \sin C$
8. $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$
9. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$
10. $\sin^2 A - \sin^2 B + \sin^2 C = 2\sin A \cos B \sin C$
11. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
12. $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$

HINTS TO SOME SELECTED QUESTIONS

7. LHS $= \sin(A+B+C-2A) + \sin(A+B+C-2B) - \sin(A+B+C-2C)$
$= \sin(\pi - 2A) + \sin(\pi - 2B) - \sin(\pi - 2C)$
$= \sin 2A + \sin 2B - \sin 2C$.

8. Multiply both sides by $2\sin A \sin B \sin C$.

12. $A+B+C=\pi \Rightarrow 2A+2B = 2\pi - 2C$
$\Rightarrow \tan(2A+2B) = \tan(2\pi - 2C) = -\tan 2C$.

Trigonometric Equations

TRIGONOMETRIC EQUATIONS

Equations which involve trigonometric functions of unknown angles are known as trigonometric equations.

For example, $\sin x = \frac{1}{2}$, $\tan x = -\sqrt{3}$, $\cos x = \sin 2x$, etc.

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

A trigonometric equation may have an infinite number of solutions.

PRINCIPAL SOLUTIONS *The solutions of a trigonometric equation for which* $0 \le x < 2\pi$ *are called its principal solutions.*

GENERAL SOLUTION *A solution of a trigonometric equation, generalised by means of periodicity, is known as the general solution.*

REMARK We shall use 'I' to denote the set of all integers.

GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

THEOREM 1 *(i)* $\sin\theta = 0 \Leftrightarrow \theta = n\pi, n \in I$

(ii) $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I$

(iii) $\tan\theta = 0 \Leftrightarrow \theta = n\pi, n \in I$

PROOF (i) $\sin\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Leftrightarrow \theta = n\pi, n \in I.$

(ii) $\cos\theta = 0 \Leftrightarrow \theta = 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Leftrightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I.$

(iii) $\tan\theta = 0 \Leftrightarrow \frac{\sin\theta}{\cos\theta} = 0$

$\Leftrightarrow \sin\theta = 0$

$\Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Leftrightarrow \theta = n\pi, n \in I.$

EXAMPLE *Find the general solution of each of the following equations:*

(i) $\sin 2x = 0$ (ii) $\sin 3x = 0$ (iii) $\sin \frac{x}{2} = 0$

(iv) $\sin\left(x - \frac{\pi}{4}\right) = 0$ (v) $\cos 2x = 0$ (vi) $\cos\left(x + \frac{\pi}{10}\right) = 0$

(vii) $\tan 3x = 0$ (viii) $\tan\left(2x + \frac{\pi}{8}\right) = 0$ (ix) $\tan\left(4x + \frac{\pi}{6}\right) = 0$

SOLUTION (i) $\sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \frac{n\pi}{2}$, where $n \in I$.

(ii) $\sin 3x = 0 \Rightarrow 3x = n\pi \Rightarrow x = \frac{n\pi}{3}$, where $n \in I$.

(iii) $\sin \frac{x}{2} = 0 \Rightarrow \frac{x}{2} = n\pi \Rightarrow x = 2n\pi$, where $n \in I$.

(iv) $\sin\left(x - \frac{\pi}{4}\right) = 0 \Rightarrow \left(x - \frac{\pi}{4}\right) = n\pi \Rightarrow x = \left(n\pi + \frac{\pi}{4}\right)$, where $n \in I$.

(v) $\cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{4}$, where $n \in I$.

(vi) $\cos\left(x + \frac{\pi}{10}\right) = 0 \Rightarrow \left(x + \frac{\pi}{10}\right) = (2n+1)\frac{\pi}{2} \Rightarrow x = \left\{(2n+1)\frac{\pi}{2} - \frac{\pi}{10}\right\}$,
where $n \in I$.

(vii) $\tan 3x = 0 \Rightarrow 3x = n\pi \Rightarrow x = \frac{n\pi}{3}$, where $n \in I$.

(viii) $\tan\left(2x + \frac{\pi}{8}\right) = 0 \Rightarrow \left(2x + \frac{\pi}{8}\right) = n\pi \Rightarrow x = \frac{1}{2}\left(n\pi - \frac{\pi}{8}\right)$,
where $n \in I$.

(ix) $\tan\left(4x + \frac{\pi}{6}\right) = 0 \Rightarrow \left(4x + \frac{\pi}{6}\right) = n\pi \Rightarrow x = \frac{1}{4}\left(n\pi - \frac{\pi}{6}\right)$,
where $n \in I$.

THEOREM 2 $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, *where* $n \in I$.

PROOF We have

$$\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$$
$$\Rightarrow 2\cos\frac{(\theta+\alpha)}{2}\sin\frac{(\theta-\alpha)}{2} = 0$$
$$\Rightarrow \cos\frac{(\theta+\alpha)}{2} = 0 \text{ or } \sin\frac{(\theta-\alpha)}{2} = 0$$
$$\Rightarrow \frac{(\theta+\alpha)}{2} = (2n+1)\frac{\pi}{2} \text{ or } \frac{(\theta-\alpha)}{2} = n\pi$$
$$\Rightarrow (\theta+\alpha) = (2n+1)\pi \text{ or } (\theta-\alpha) = 2n\pi$$
$$\Rightarrow \theta = [(2n+1)\pi - \alpha] \text{ or } \theta = (2n\pi + \alpha)$$

$$\Rightarrow \ \theta = [(\text{an odd multiple of } \pi) - \alpha]$$

$$\text{or } \theta = [(\text{an even multiple of } \pi) + \alpha]$$

$$\Rightarrow \ \theta = n\pi + (-1)^n \alpha, \text{ where } n \in I.$$

$$\therefore \quad \sin\theta = \sin\alpha \ \Rightarrow \ \theta = n\pi + (-1)^n\alpha, \text{ where } n \in I.$$

THEOREM 3 $\cos\theta = \cos\alpha \ \Rightarrow \ \theta = 2n\pi \pm \alpha$, *where* $n \in I$.

PROOF We have

$$\cos\theta = \cos\alpha \ \Rightarrow \ \cos\theta - \cos\alpha = 0$$

$$\Rightarrow \ -2\sin\frac{(\theta+\alpha)}{2}\sin\frac{(\theta-\alpha)}{2} = 0$$

$$\Rightarrow \ \sin\frac{(\theta+\alpha)}{2} = 0 \ \text{ or } \ \sin\frac{(\theta-\alpha)}{2} = 0$$

$$\Rightarrow \ \frac{(\theta+\alpha)}{2} = n\pi \ \text{ or } \ \frac{(\theta-\alpha)}{2} = n\pi$$

$$\Rightarrow \ (\theta+\alpha) = 2n\pi \ \text{ or } \ (\theta-\alpha) = 2n\pi$$

$$\Rightarrow \ \theta = (2n\pi - \alpha) \ \text{ or } \ \theta = (2n\pi + \alpha)$$

$$\Rightarrow \ \theta = (2n\pi \pm \alpha).$$

$$\therefore \quad \cos\theta = \cos\alpha \ \Rightarrow \ \theta = (2n\pi \pm \alpha), \text{ where } n \in I.$$

THEOREM 4 *If* θ *and* α *are not odd multiples of* $\frac{\pi}{2}$ *then*

$$\tan\theta = \tan\alpha \ \Rightarrow \ \theta = n\pi + \alpha, \textit{ where } n \in I.$$

PROOF We have

$$\tan\theta = \tan\alpha \ \Rightarrow \ \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \ \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\Rightarrow \ \sin(\theta - \alpha) = 0$$

$$\Rightarrow \ (\theta - \alpha) = n\pi, \text{ where } n \in I$$

$$\Rightarrow \ \theta = (n\pi + \alpha), \text{ where } n \in I$$

$$\therefore \quad \tan\theta = \tan\alpha \ \Rightarrow \ \theta = (n\pi + \alpha), \text{ where } n \in I.$$

THEOREM 5 *(i)* $\sin^2\theta = \sin^2\alpha \ \Leftrightarrow \ \theta = n\pi \pm \alpha$

(ii) $\cos^2\theta = \cos^2\alpha \ \Leftrightarrow \ \theta = n\pi \pm \alpha$

(iii) $\tan^2\theta = \tan^2\alpha \ \Leftrightarrow \ \theta = n\pi \pm \alpha$

PROOF (i) $\sin^2\theta = \sin^2\alpha$

$$\Rightarrow \ \frac{(1-\cos 2\theta)}{2} = \frac{(1-\cos 2\alpha)}{2}$$

$$\Rightarrow \ \cos 2\theta = \cos 2\alpha$$

$$\Rightarrow \ 2\theta = 2n\pi \pm 2\alpha$$

$$\Rightarrow \ \theta = (n\pi \pm \alpha), \ n \in I.$$

(ii) $\cos^2\theta = \cos^2\alpha$

$$\Rightarrow \quad \frac{(1 + \cos 2\theta)}{2} = \frac{(1 + \cos 2\alpha)}{2}$$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha$

$\Rightarrow \quad \theta = (n\pi \pm \alpha)$, where $n \in I$.

(iii) $\tan^2\theta = \tan^2\alpha$

$$\Rightarrow \quad \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha$, where $n \in I$

$\Rightarrow \quad \theta = (n\pi \pm \alpha)$, where $n \in I$.

THEOREM 6 $\;a\cos\theta + b\sin\theta = c \;\Leftrightarrow\; \theta = 2n\pi + \alpha \pm \beta$,

where $\cos\alpha = \dfrac{a}{\sqrt{a^2 + b^2}}$, $\cos\beta = \dfrac{c}{\sqrt{a^2 + b^2}}$ *and* $\sin\alpha = \dfrac{b}{\sqrt{a^2 + b^2}}$.

PROOF The given equation is $a\cos\theta + b\sin\theta = c$. ... (i)

Dividing (i) throughout by $\sqrt{a^2 + b^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2}} \cdot \cos\theta + \frac{b}{\sqrt{a^2 + b^2}} \cdot \sin\theta = \frac{c}{\sqrt{a^2 + b^2}}$$

$\Rightarrow \quad \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos\beta$,

where $\cos\alpha = \dfrac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \dfrac{b}{\sqrt{a^2 + b^2}}$ and $\cos\beta = \dfrac{c}{\sqrt{a^2 + b^2}}$

$\Rightarrow \quad \cos(\theta - \alpha) = \cos\beta$

$\Rightarrow \quad (\theta - \alpha) = 2n\pi \pm \beta$

$\Rightarrow \quad \theta = (2n\pi + \alpha \pm \beta)$.

SUMMARY

1. (i) $\sin\theta = 0 \Rightarrow \theta = n\pi$
(ii) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\dfrac{\pi}{2}$
(iii) $\tan\theta = 0 \Rightarrow \theta = n\pi$
where $n \in I$

2. (i) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$
(ii) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$
(iii) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$
where $n \in I$

3. (i) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
(ii) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
(iii) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
where $n \in I$

SOLVED EXAMPLES

EXAMPLE 1 *Find the principal solutions of each of the following equations:*

(i) $\sin x = \frac{1}{2}$ *(ii)* $\cos x = \frac{1}{\sqrt{2}}$ *(iii)* $\tan x = \frac{1}{\sqrt{3}}$

SOLUTION (i) The given equation is $\sin x = \frac{1}{2}$.

We know that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$.

$\therefore \sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$.

Hence, the principal solutions are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

(ii) The given equation is $\cos x = \frac{1}{\sqrt{2}}$.

We know that $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\left(2\pi - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$\therefore \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$.

Hence, the principal solutions are $x = \frac{\pi}{4}$ and $x = \frac{7\pi}{4}$.

(iii) The given equation is $\tan x = \frac{1}{\sqrt{3}}$.

We know that $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\tan\left(\pi + \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

$\therefore \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\tan \frac{7\pi}{6} = \frac{1}{\sqrt{3}}$.

Hence, the principal solutions are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.

EXAMPLE 2 *Find the principal solutions of each of the following equations:*

(i) $\sin x = \frac{-\sqrt{3}}{2}$ *(ii)* $\cos x = \frac{-1}{2}$ *(iii)* $\cot x = -\sqrt{3}$

SOLUTION (i) The given equation is $\sin x = \frac{-\sqrt{3}}{2}$.

We know that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$\therefore \sin\left(\pi + \frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = \frac{-\sqrt{3}}{2}$, and $\left(2\pi - \frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = \frac{-\sqrt{3}}{2}$.

$\therefore \sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$ and $\sin \frac{5\pi}{3} = \frac{-\sqrt{3}}{2}$.

Hence, the principal solutions are $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$.

(ii) The given equation is $\cos x = \frac{-1}{2}$.

We know that $\cos \frac{\pi}{3} = \frac{1}{2}$.

$\therefore \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$, and $\cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$.

$\therefore \cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\cos\frac{4\pi}{3} = -\frac{1}{2}$.

Hence, the principal solutions are $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.

(iii) The given equation is $\cot x = -\sqrt{3} \Leftrightarrow \tan x = -\frac{1}{\sqrt{3}}$.

We know that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

$\therefore \tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = \frac{-1}{\sqrt{3}}$, and $\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = \frac{-1}{\sqrt{3}}$.

$\therefore \tan\frac{5\pi}{6} = \frac{-1}{\sqrt{3}}$ and $\tan\frac{11\pi}{6} = \frac{-1}{\sqrt{3}}$.

Hence, the principal solutions are $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$.

EXAMPLE 3 *In each of the following, find the general value of x satisfying the equation:*

(i) $\sin x = \frac{1}{\sqrt{2}}$ *(ii)* $\cos x = \frac{1}{2}$ *(iii)* $\tan x = \frac{1}{\sqrt{3}}$

SOLUTION (i) Given: $\sin x = \frac{1}{\sqrt{2}}$.

The least value of x in $[0, 2\pi[$ for which $\sin x = \frac{1}{\sqrt{2}}$ is $x = \frac{\pi}{4}$.

$\therefore \sin x = \sin\frac{\pi}{4} \Rightarrow x = n\pi + (-1)^n \cdot \frac{\pi}{4}$, where $n \in I$.

Hence, the general solution is $x = n\pi + (-1)^n \cdot \frac{\pi}{4}$, where $n \in I$.

(ii) Given: $\cos x = \frac{1}{2}$.

The least value of x in $[0, 2\pi[$ for which $\cos x = \frac{1}{2}$ is $x = \frac{\pi}{3}$.

$\therefore \cos x = \cos\frac{\pi}{3} \Rightarrow x = \left(2n\pi \pm \frac{\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(2n\pi \pm \frac{\pi}{3}\right)$, where $n \in I$.

(iii) Given: $\tan x = \frac{1}{\sqrt{3}}$.

The least value of x in $[0, 2\pi[$ for which $\tan x = \frac{1}{\sqrt{3}}$ is $\frac{\pi}{6}$.

$\therefore \tan x = \tan \frac{\pi}{6} \Rightarrow x = \left(n\pi + \frac{\pi}{6}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi + \frac{\pi}{6}\right)$, where $n \in I$.

EXAMPLE 4 *Find the general value of x for which* $\sqrt{3}\operatorname{cosec} x = 2$.

SOLUTION Given: $\sqrt{3}\operatorname{cosec} x = 2 \Rightarrow \operatorname{cosec} x = \frac{2}{\sqrt{3}} \Rightarrow \sin x = \frac{\sqrt{3}}{2}$.

The least value of x in $[0, 2\pi[$ for which $\sin x = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

$\therefore \quad \sin x = \sin \frac{\pi}{3} \Rightarrow x = \left\{n\pi + (-1)^n \cdot \frac{\pi}{3}\right\}$, where $n \in I$.

Hence, the general solution is $x = \left\{n\pi + (-1)^n \cdot \frac{\pi}{3}\right\}$, where $n \in I$.

EXAMPLE 5 *In each of the following, find the general value of x satisfying the equation:*

(i) $\sin x = \frac{-\sqrt{3}}{2}$ *(ii)* $\cos x = \frac{-1}{2}$ *(iii)* $\cot x = -\sqrt{3}$

SOLUTION (i) $\sin x = \frac{-\sqrt{3}}{2} = -\sin \frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin \frac{4\pi}{3}$.

$\therefore \sin x = \sin \frac{4\pi}{3} \Rightarrow x = \left\{n\pi + (-1)^n \cdot \frac{4\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left\{n\pi + (-1)^n \cdot \frac{4\pi}{3}\right)$, where $n \in I$.

(ii) $\cos x = \frac{-1}{2} = -\cos \frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos \frac{2\pi}{3}$.

$\therefore \cos x = \cos \frac{2\pi}{3} \Rightarrow x = \left(2n\pi \pm \frac{2\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(2n\pi \pm \frac{2\pi}{3}\right)$, where $n \in I$.

(iii) $\cot x = -\sqrt{3}$

$\Rightarrow \tan x = \frac{-1}{\sqrt{3}} = -\tan \frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan \frac{5\pi}{6}$

$\Rightarrow \tan x = \tan \frac{5\pi}{6}$

$\Rightarrow x = \left(n\pi + \frac{5\pi}{6}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi + \frac{5\pi}{6}\right)$, where $n \in I$.

EXAMPLE 6 *Find the general solution of each equation:*

(i) $\sqrt{3}\cot x + 1 = 0$ *(ii)* $\operatorname{cosec} x + \sqrt{2} = 0$

SOLUTION (i) $\sqrt{3}\cot x + 1 = 0$

$\Rightarrow \cot x = \frac{-1}{\sqrt{3}}$

$\Rightarrow \tan x = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) = \tan\frac{2\pi}{3}$

$\Rightarrow \tan x = \tan\frac{2\pi}{3}$

$\Rightarrow x = \left(n\pi + \frac{2\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi + \frac{2\pi}{3}\right)$, where $n \in I$.

(ii) $\operatorname{cosec} x + \sqrt{2} = 0$

$\Rightarrow \sin x = -\frac{1}{\sqrt{2}} = -\sin\frac{\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}$

$\Rightarrow \sin x = \sin\frac{5\pi}{4}$

$\Rightarrow x = \left\{n\pi + (-1)^n \cdot \frac{5\pi}{4}\right\}$, where $n \in I$.

Hence, the general solution is $x = \left\{n\pi + (-1)^n \cdot \frac{5\pi}{4}\right\}$, where $n \in I$.

EXAMPLE 7 *Find the general solution of each of the equations:*

(i) $\sin 2x = -\frac{1}{2}$ *(ii)* $\tan 3x = -1$

SOLUTION (i) $\sin 2x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\Rightarrow \sin 2x = \sin\frac{7\pi}{6}$

$\Rightarrow 2x = \left\{n\pi + (-1)^n \cdot \frac{7\pi}{6}\right\}$, where $n \in I$

$\Rightarrow x = \left\{\frac{n\pi}{2} + (-1)^n \cdot \frac{7\pi}{12}\right\}$, where $n \in I$.

Hence, the general solution is $x = \left\{\frac{n\pi}{2} + (-1)^n \cdot \frac{7\pi}{12}\right\}$, where $n \in I$.

(ii) $\tan 3x = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$\Rightarrow \tan 3x = \tan\frac{3\pi}{4}$

$$\Rightarrow\ 3x = \left(n\pi + \frac{3\pi}{4}\right), \text{ where } n \in I.$$

$$\Rightarrow\ x = \left(\frac{n\pi}{3} + \frac{\pi}{4}\right), \text{ where } n \in I.$$

Hence, the general solution is $x = \left(\frac{n\pi}{3} + \frac{\pi}{4}\right)$, where $n \in I$.

EXAMPLE 8 *Find the general solution of each of the equations:*

(i) $4\sin^2 x = 1$ *(ii)* $2\cos^2 x = 1$ *(iii)* $\cot^2 x = 3$

SOLUTION (i) $4\sin^2 x = 1 \Rightarrow \sin^2 x = \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \sin^2\frac{\pi}{6}$

$$\Rightarrow\ \sin^2 x = \sin^2\frac{\pi}{6}$$

$$\Rightarrow\ x = \left\{n\pi \pm \frac{\pi}{6}\right\}, \text{ where } n \in I.$$

Hence, the general solution is $x = \left(n\pi \pm \frac{\pi}{6}\right)$, $n \in I$.

(ii) $2\cos^2 x = 1 \Rightarrow \cos^2 x = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\frac{\pi}{4}$

$$\Rightarrow\ \cos^2 x = \cos^2\frac{\pi}{4}$$

$$\Rightarrow\ x = \left(n\pi \pm \frac{\pi}{4}\right), \text{ where } n \in I.$$

Hence, the general solution is $x = \left(n\pi \pm \frac{\pi}{4}\right)$, $n \in I$.

(iii) $\cot^2 x = 3 \Rightarrow \tan^2 x = \frac{1}{3} = \left(\frac{1}{\sqrt{3}}\right)^2 = \tan^2\frac{\pi}{6}$

$$\Rightarrow\ \tan^2 x = \tan^2\frac{\pi}{6}$$

$$\Rightarrow\ x = \left(n\pi \pm \frac{\pi}{6}\right), \text{ where } n \in I.$$

Hence, the general solution is $\left(n\pi \pm \frac{\pi}{6}\right)$, where $n \in I$.

EXAMPLE 9 *Find the general solution of the equation* $\sin 2x + \sin 4x + \sin 6x = 0$.

SOLUTION The given equation may be written as $(\sin 6x + \sin 2x) + \sin 4x = 0$

$$\Rightarrow\ 2\sin\frac{(6x + 2x)}{2}\cos\frac{(6x - 2x)}{2} + \sin 4x = 0$$

$$\left[\because\ \sin C + \sin D = 2\sin\frac{(C + D)}{2}\cos\frac{(C - D)}{2}\right]$$

$\Rightarrow \quad 2\sin 4x \cos 2x + \sin 4x = 0$

$\Rightarrow \quad \sin 4x(2\cos 2x + 1) = 0$

$\Rightarrow \quad \sin 4x = 0 \text{ or } 2\cos 2x + 1 = 0$

$\Rightarrow \quad \sin 4x = 0 \text{ or } \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\Rightarrow \quad \sin 4x = 0 \text{ or } \cos 2x = \cos\frac{2\pi}{3}$

$\Rightarrow \quad 4x = n\pi \text{ or } 2x = \left(2m\pi \pm \frac{2\pi}{3}\right)$, where $m, n \in I$

$\Rightarrow \quad x = \frac{n\pi}{4} \text{ or } x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$.

Hence, the general solution is given by $x = \frac{n\pi}{4}$ or $x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$.

EXAMPLE 10 *Solve:* $2\cos^2 x + 3\sin x = 0$.

SOLUTION We have

$2\cos^2 x + 3\sin x = 0$

$\Rightarrow \quad 2(1 - \sin^2 x) + 3\sin x = 0$

$\Rightarrow \quad 2\sin^2 x - 3\sin x - 2 = 0$

$\Rightarrow \quad 2\sin^2 x - 4\sin x + \sin x - 2 = 0$

$\Rightarrow \quad 2\sin x(\sin x - 2) + (\sin x - 2) = 0$

$\Rightarrow \quad (\sin x - 2)(2\sin x + 1) = 0$

$\Rightarrow \quad (\sin x - 2) = 0 \text{ or } (2\sin x + 1) = 0$

$\Rightarrow \quad \sin x = 2 \text{ or } (2\sin x + 1) = 0$

$\Rightarrow \quad 2\sin x + 1 = 0 \quad [\because \sin x = 2 \text{ is not possible}]$

$\Rightarrow \quad \sin x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\Rightarrow \quad \sin x = \sin\frac{7\pi}{6}$

$\Rightarrow \quad x = \left\{n\pi + (-1)^n \cdot \frac{7\pi}{6}\right\}$, where $n \in I$.

Hence, the general solution is given by $x = \left\{n\pi + (-1)^n \cdot \frac{7\pi}{6}\right\}$, where $n \in I$.

EXAMPLE 11 *Find the general solution for each of the following equations:*

(i) $\cos 4x = \cos 2x$ *(ii)* $\cos 3x = \sin 2x$

(iii) $\sin 3x + \cos 2x = 0$ *(iv)* $\sin mx + \sin nx = 0$

SOLUTION (i) $\cos 4x = \cos 2x$

$\Rightarrow \cos 4x - \cos 2x = 0$

$$\Rightarrow -2\sin\frac{(4x+2x)}{2}\sin\frac{(4x-2x)}{2}=0$$

$$\left[\because \cos C-\cos D=-2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}\right]$$

$$\Rightarrow -2\sin 3x\sin x=0$$

$$\Rightarrow \sin 3x=0 \text{ or } \sin x=0$$

$$\Rightarrow 3x=n\pi \text{ or } x=m\pi, \text{where } m, n\in I$$

$$\Rightarrow x=\frac{n\pi}{3} \text{ or } x=m\pi, \text{where } m, n\in I.$$

Hence, the general solution is $x=\frac{n\pi}{3}$ or $x=m\pi$, where $m, n\in I$.

(ii) $\cos 3x=\sin 2x$

$$\Rightarrow \cos 3x=\cos\left(\frac{\pi}{2}-2x\right)$$

$$\Rightarrow 3x=2n\pi\pm\left(\frac{\pi}{2}-2x\right) \quad [\cos\theta=\cos\alpha \Rightarrow \theta=2n\pi\pm\alpha]$$

$$\Rightarrow 3x=2n\pi+\left(\frac{\pi}{2}-2x\right) \text{ or } 3x=2n\pi-\left(\frac{\pi}{2}-2x\right), \text{where } n\in I$$

$$\Rightarrow 5x=2n\pi+\frac{\pi}{2} \text{ or } x=\left(2n\pi-\frac{\pi}{2}\right), \text{where } n\in I$$

$$\Rightarrow x=\left(\frac{2n\pi}{5}+\frac{\pi}{10}\right) \text{ or } x=\left(2n\pi-\frac{\pi}{2}\right), \text{where } n\in I.$$

Hence, the general solution is $x=\left(\frac{2n\pi}{5}+\frac{\pi}{10}\right)$ or $x=\left(2n\pi-\frac{\pi}{2}\right)$, where $n\in I$.

(iii) $\sin 3x+\cos 2x=0$

$$\Rightarrow \cos 2x=-\sin 3x=\cos\left(\frac{\pi}{2}+3x\right)$$

$$\Rightarrow \cos 2x=\cos\left(\frac{\pi}{2}+3x\right)$$

$$\Rightarrow 2x=2n\pi\pm\left(\frac{\pi}{2}+3x\right), \text{where } n\in I$$

$$\Rightarrow 2x=2n\pi+\left(\frac{\pi}{2}+3x\right) \text{ or } 2x=2n\pi-\left(\frac{\pi}{2}+3x\right), \text{where } n\in I$$

$$\Rightarrow x=\left(-2n\pi-\frac{\pi}{2}\right) \text{ or } x=\left(\frac{2n\pi}{5}-\frac{\pi}{10}\right), \text{where } n\in I$$

$$\Rightarrow x=\left(2n\pi-\frac{\pi}{2}\right) \text{ or } x=\left(\frac{2n\pi}{5}-\frac{\pi}{10}\right), \text{where } n\in I.$$

Hence, the general solution is $x = \left(2n\pi - \frac{\pi}{2}\right)$ or $x = \left(\frac{2n\pi}{5} - \frac{\pi}{10}\right)$,

where $n \in I$.

NOTE $\left(-2n\pi - \frac{\pi}{2}\right)$ and $\left(2n\pi - \frac{\pi}{2}\right)$ give the same result as $n \in I$.

(iv) $\sin mx + \sin nx = 0$

$$\Rightarrow \quad 2\sin\frac{(m+n)x}{2}\cos\frac{(m-n)x}{2} = 0$$

$$\text{or } \sin\frac{(m+n)x}{2} = 0 \quad \text{or} \quad \cos\frac{(m-n)x}{2} = 0$$

$$\Rightarrow \quad \frac{(m+n)x}{2} = p\pi \text{ or } \frac{(m-n)x}{2} = (2q+1)\frac{\pi}{2}, \text{ where } p, q \in I$$

$$\Rightarrow \quad x = \frac{2p\pi}{(m+n)} \text{ or } x = \frac{(2q+1)\pi}{(m-n)}, \text{ where } p, q \in I.$$

Hence, the general solution is $x = \frac{2p\pi}{(m+n)}$ or $x = \frac{(2q+1)\pi}{(m-n)}$,

where $p, q \in I$.

EXAMPLE 12 *Solve:* $\sqrt{3}\cos x - \sin x = 1$.

SOLUTION Given: $\sqrt{3}\cos x - \sin x = 1$. ... (i)

Dividing both sides of (i) by $\sqrt{(\sqrt{3})^2 + (-1)^2}$, i.e., by 2, we get

$$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\left(x + \frac{\pi}{6}\right) = \cos\frac{\pi}{3} \quad \left[\because \frac{1}{2} = \cos\frac{\pi}{3}\right]$$

$$\Rightarrow \quad \left(x + \frac{\pi}{6}\right) = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in I \; [\because \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$$

$$\Rightarrow \quad \left(x + \frac{\pi}{6}\right) = \left(2n\pi + \frac{\pi}{3}\right) \text{ or } \left(x + \frac{\pi}{6}\right) = \left(2n\pi - \frac{\pi}{3}\right),$$

$$x = \left(2n\pi + \frac{\pi}{6}\right) \text{ or } x = \left(2n\pi - \frac{\pi}{2}\right), \text{ where } n \in I.$$

Hence, the general solution is $x = \left(2n\pi + \frac{\pi}{6}\right)$ or $x = \left(2n\pi - \frac{\pi}{2}\right)$,

where $n \in I$.

EXAMPLE 13 *Solve:* $\sec x - \tan x = \sqrt{3}$.

SOLUTION We have

$$\sec x - \tan x = \sqrt{3} \Rightarrow \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \sqrt{3} \Rightarrow (1 - \sin x) = \sqrt{3}\cos x$$

$$\Rightarrow \sqrt{3}\cos x + \sin x = 1.$$

Thus, the given equation becomes

$$\sqrt{3}\cos x + \sin x = 1. \qquad \ldots \text{(i)}$$

Dividing both sides of (i) by $\sqrt{(\sqrt{3})^2 + 1^2}$, i.e., by 2, we get

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow \left(x - \frac{\pi}{6}\right) = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in I \; [\because \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$$

$$\Rightarrow \left(x - \frac{\pi}{6}\right) = \left(2n\pi + \frac{\pi}{3}\right) \text{ or } \left(x - \frac{\pi}{6}\right) = \left(2n\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow x = 2n\pi + \left(+\frac{\pi}{3} + \frac{\pi}{6}\right) \text{ or } x = 2n\pi + \left(-\frac{\pi}{3} + \frac{\pi}{6}\right), \text{ where } n \in I$$

$$\Rightarrow x = \left(2n\pi + \frac{\pi}{2}\right) \text{ or } x = \left(2n\pi - \frac{\pi}{6}\right), \text{ where } n \in I$$

$$\Rightarrow x = \left(2n\pi - \frac{\pi}{6}\right), \text{ where } n \in I$$

$$\left[\because \sec x \text{ is not defined when } x = \left(2n\pi + \frac{\pi}{2}\right)\right].$$

Hence, the general solution is $x = \left(2n\pi - \frac{\pi}{6}\right)$, where $n \in I$.

EXERCISE 17

1. Find the principal solutions of each of the following equations:

(i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $\cos x = \frac{1}{2}$ (iii) $\tan x = \sqrt{3}$

(iv) $\cot x = \sqrt{3}$ (v) $\operatorname{cosec} x = 2$ (vi) $\sec x = \frac{2}{\sqrt{3}}$

2. Find the principal solutions of each of the following equations:

(i) $\sin x = \frac{-1}{2}$ (ii) $\sqrt{2}\cos x + 1 = 0$

(iii) $\tan x = -1$ (iv) $\sqrt{3}\operatorname{cosec} x + 2 = 0$

(v) $\tan x = -\sqrt{3}$ (vi) $\sqrt{3}\sec x + 2 = 0$

Find the general solution of each of the following equations:

3. (i) $\sin 3x = 0$ (ii) $\sin\frac{3x}{2} = 0$ (iii) $\sin\left(x + \frac{\pi}{5}\right) = 0$

(iv) $\cos 2x = 0$ (v) $\cos\frac{5x}{2} = 0$ (vi) $\cos\left(x + \frac{\pi}{10}\right) = 0$

(vii) $\tan 2x = 0$ (viii) $\tan\left(3x + \frac{\pi}{6}\right) = 0$ (ix) $\tan\left(2x - \frac{\pi}{4}\right) = 0$

4. (i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $\cos x = 1$ (iii) $\sec x = \sqrt{2}$

5. (i) $\cos x = \frac{-1}{2}$ (ii) $\operatorname{cosec} x = -\sqrt{2}$ (iii) $\tan x = -1$

6. (i) $\sin 2x = \frac{1}{2}$ (ii) $\cos 3x = \frac{1}{\sqrt{2}}$ (iii) $\tan \frac{2x}{3} = \sqrt{3}$

7. (i) $\sec 3x = -2$ (ii) $\cot 4x = -1$ (iii) $\operatorname{cosec} 3x = \frac{-2}{\sqrt{3}}$

8. (i) $4\cos^2 x = 1$ (ii) $4\sin^2 x - 3 = 0$ (iii) $\tan^2 x = 1$

9. (i) $\cos 3x = \cos 2x$ (ii) $\cos 5x = \sin 3x$ (iii) $\cos mx = \sin nx$

10. $\sin x = \tan x$

11. $4\sin x \cos x + 2\sin x + 2\cos x + 1 = 0$

12. $\sec^2 2x = 1 - \tan 2x$

13. $\tan^3 x - 3\tan x = 0$

14. $\sin x + \sin 3x + \sin 5x = 0$

15. $\sin x \tan x - 1 = \tan x - \sin x$

16. $\cos x + \sin x = 1$

17. $\cos x - \sin x = -1$

18. $\sqrt{3}\cos x + \sin x = 1$

19. $2\tan x - \cot x + 1 = 0$

20. $\sin x \tan x - 1 = \tan x - \sin x$

21. $\cot x + \tan x = 2\operatorname{cosec} x$

ANSWERS (EXERCISE 17)

1. (i) $\frac{\pi}{3}, \frac{2\pi}{3}$ (ii) $\frac{\pi}{3}, \frac{5\pi}{3}$ (iii) $\frac{\pi}{3}, \frac{4\pi}{3}$

(iv) $\frac{\pi}{6}, \frac{7\pi}{6}$ (v) $\frac{\pi}{6}, \frac{5\pi}{6}$ (vi) $\frac{\pi}{6}, \frac{11\pi}{6}$

2. (i) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (ii) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (iii) $\frac{3\pi}{4}, \frac{7\pi}{4}$

(iv) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (v) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (vi) $\frac{5\pi}{6}, \frac{7\pi}{6}$

3. (i) $\frac{n\pi}{3}, n \in I$ (ii) $\frac{2n\pi}{3}, n \in I$ (iii) $\left(n\pi - \frac{\pi}{5}\right), n \in I$

(iv) $(2n+1)\frac{\pi}{4}, n \in I$ (v) $(2n+1)\frac{\pi}{5}, n \in I$ (vi) $\left(n\pi + \frac{2\pi}{5}\right), n \in I$

(vii) $\frac{n\pi}{2}, n \in I$ (viii) $\left(\frac{n\pi}{3} - \frac{\pi}{18}\right), n \in I$ (ix) $\left(\frac{n\pi}{2} + \frac{\pi}{8}\right), n \in I$

4. (i) $x = n\pi + (-1)^n \cdot \frac{\pi}{3}, n \in I$ (ii) $x = 2n\pi, n \in I$

(iii) $x = \left(2n\pi \pm \frac{\pi}{4}\right), n \in I$

5. (i) $x = \left(2\pi \pm \frac{2\pi}{3}\right), n \in I$ (ii) $x = n\pi + (-1)^n \cdot \frac{5\pi}{4}, n \in I$

(iii) $x = \left(n\pi + \frac{3\pi}{4}\right), n \in I$

6. (i) $x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{12}, n \in I$ (ii) $x = \left(\frac{2n\pi}{3} \pm \frac{\pi}{12}\right), n \in I$

(iii) $x = (3n + 1)\frac{\pi}{2}, n \in I$

7. (i) $x = \left(\frac{2n\pi}{3} \pm \frac{2\pi}{9}\right), n \in I$ (ii) $x = (4n + 3)\frac{\pi}{16}, n \in I$

(iii) $x = \frac{n\pi}{3} + (-1)^n \cdot \frac{4\pi}{9}, n \in I$

8. (i) $x = \left(n\pi \pm \frac{\pi}{3}\right), n \in I$ (ii) $x = \left(n\pi \pm \frac{\pi}{3}\right), n \in I$

(iii) $x = \left(n\pi \pm \frac{\pi}{4}\right), n \in I$

9. (i) $x = 2n\pi$ or $x = \frac{2n\pi}{5}$, where $n \in I$

(ii) $x = \left(\frac{n\pi}{4} + \frac{\pi}{16}\right)$ or $x = \left(n\pi - \frac{\pi}{4}\right)$, where $n \in I$

(iii) $x = \frac{(4k + 1)\pi}{2(m + n)}$ or $x = \frac{(4k - 1)\pi}{2(m - n)}$, where $k \in I$

10. $x = n\pi$ or $x = 2m\pi$, where $m, n \in I$

11. $x = \left(2n\pi \pm \frac{2\pi}{3}\right)$ or $x = m\pi + (-1)^m \cdot \frac{7\pi}{6}$, where $m, n \in I$

12. $x = \frac{n\pi}{2}$ or $x = \left(\frac{m\pi}{2} + \frac{3\pi}{8}\right)$, where $m, n \in I$

13. $x = n\pi$ or $x = \left(m\pi + \frac{\pi}{3}\right)$ or $x = \left(p\pi + \frac{2\pi}{3}\right)$, where $m, n, p \in I$

14. $x = \frac{n\pi}{3}$ or $x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$

15. $x = n\pi + (-1)^n \cdot \frac{\pi}{2}$ or $x = \left(m\pi + \frac{3\pi}{4}\right)$, where $m, n \in I$

16. $x = \left(2n\pi + \frac{\pi}{2}\right)$ or $x = 2n\pi$, where $n \in I$

17. $x = \left(2n\pi + \frac{\pi}{2}\right)$ or $x = (2n - 1)\pi$, where $n \in I$

18. $x = \left(2n\pi + \frac{\pi}{2}\right)$ or $x = \left(2n\pi - \frac{\pi}{6}\right)$, where $n \in I$

19. $x = n\pi + \frac{3\pi}{4}$ or $x = m\pi + \tan^{-1}\frac{1}{2}$, where $m, n \in I$

20. $x = \left(m\pi - \frac{\pi}{4}\right)$ or $x = n\pi + (-1)^n \frac{\pi}{2}$, where $m, n \in I$

21. $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in I$

HINTS TO SOME SELECTED QUESTIONS

10. $\sin x = \tan x \Rightarrow \sin x = \frac{\sin x}{\cos x}$

$\Rightarrow \sin x \cos x - \sin x = 0$

$\Rightarrow \sin x(\cos x - 1) = 0$

$\Rightarrow \sin x = 0 \text{ or } \cos x = 1$

$\Rightarrow x = n\pi \text{ or } x = 2m\pi$, where $m, n \in I$.

11. $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$

$\Rightarrow 2\sin x(2\cos x + 1) + (2\cos x + 1) = 0$

$\Rightarrow (2\cos x + 1)(2\sin x + 1) = 0$

$\Rightarrow \cos x = -\frac{1}{2} \text{ or } \sin x = -\frac{1}{2}$

$\Rightarrow \cos x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

or $\sin x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\Rightarrow x = \left(2n\pi \pm \frac{2\pi}{3}\right) \text{ or } x = m\pi + (-1)^m \cdot \frac{7\pi}{6}$, where $m, n \in I$.

12. $\sec^2 2x = 1 - \tan 2x$

$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \tan^2 2x + \tan 2x = 0$

$\Rightarrow \tan 2x(\tan 2x + 1) = 0$

$\Rightarrow \tan 2x = 0 \text{ or } \tan 2x = -1$

$\Rightarrow 2x = n\pi \text{ or } \tan 2x = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$\Rightarrow x = \frac{n\pi}{2} \text{ or } 2x = m\pi + \frac{3\pi}{4}$, where $m, n \in I$

$\Rightarrow \quad x = \frac{n\pi}{2}$ or $x = \frac{m\pi}{2} + \frac{3\pi}{8}$, where $m, n \in I$.

13. $\tan^3 x - 3\tan x = 0 \Rightarrow \tan x(\tan^2 x - 3) = 0$

$\Rightarrow \tan x = 0$ or $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}$

$\Rightarrow x = n\pi$ or $\tan x = \tan\frac{\pi}{3}$ or $\tan x = -\tan\frac{\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right)$

$\Rightarrow x = n\pi$ or $x = m\pi + \frac{\pi}{3}$ or $x = p\pi + \frac{2\pi}{3}$, where $m, n, p \in I$.

15. The given equation becomes

$(\sin x - 1)(\sin x + \cos x) = 0$

$\Rightarrow \quad \sin x = 1$ or $\tan x = -1$

$\Rightarrow \quad \sin x = \sin\frac{\pi}{2}$ or $\tan x = \tan\frac{3\pi}{4}$

$\Rightarrow \quad x = n\pi + (-1)^n\frac{\pi}{2}$ or $x = m\pi + \frac{3\pi}{4}$, where $m, n \in I$.

16. $\cos x + \sin x = 1 \Rightarrow \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow x - \frac{\pi}{4} = 2n\pi + \frac{\pi}{4}$ or $x - \frac{\pi}{4} = 2n\pi - \frac{\pi}{4}$

$\Rightarrow x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi$, where $n \in I$.

17. $\cos x - \sin x = -1$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \frac{-1}{\sqrt{2}}$

$\Rightarrow \quad \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right)$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$

$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$

$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi + \frac{3\pi}{4}$ or $x + \frac{\pi}{4} = 2n\pi - \frac{3\pi}{4}$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2}$ or $x = (2n - 1)\pi$, where $n \in I$.

19. The given equation is

$$2\tan^2 x + \tan x - 1 = 0$$

$\Rightarrow$ $(\tan x + 1)(2\tan x - 1) = 0$

$\Rightarrow$ $\tan x = -1$ or $\tan x = \frac{1}{2}$

$\Rightarrow$ $\tan x = \tan\frac{3\pi}{4}$ or $\tan x = \tan\left(\tan^{-1}\frac{1}{2}\right)$

$\Rightarrow$ $x = n\pi + \frac{3\pi}{4}$ or $x = m\pi + \tan^{-1}\frac{1}{2}$, where $m, n \in I$.

20. The given equation is

$$\sin x \tan x + \sin x - 1 - \tan x = 0$$

$\Rightarrow$ $\sin x(\tan x + 1) - (\tan x + 1) = 0$

$\Rightarrow$ $(\tan x + 1)(\sin x - 1) = 0$

$\Rightarrow$ $\tan x = -1 = \tan\left(-\frac{\pi}{4}\right)$ or $\sin x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow$ $x = \left(m\pi - \frac{\pi}{4}\right)$ or $x = n\pi + (-1)^n\frac{\pi}{2}$.

21. The given equation is

$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{2}{\sin x} \Rightarrow \cos^2 x + \sin^2 x = 2\cos x$

$\Rightarrow$ $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow$ $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in I$.